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The neutral curves for periodic perturbations of finite amplitude of plane Poiseuille flow

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It is shown that there exist undamped solutions for perturbations of finite amplitude of plane Poiseuille flow, which are periodic in the direction of the axis of the channel. The shift in the 'neutral curve' as a function of the amplitude λ^* of the disturbance is shown in figure 2. The solution is obtained by a perturbation method in which the eigenfunctions and the eigenvalue c are expanded in power series of the amplitude λ, as shown in (14), (15), (16) and (17). Near the neutral curve for a finite amplitude disturbance, the curvature of the mean flow shows a tendency to become negative (figure 5).

Analytical formulation

We wish to determine whether the Navier–Stokes equation,

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \nabla^2 \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial y} = \frac{1}{R}\nabla^4 \psi, \tag{1}$$

for the stream function ψ in two-dimensional flow, has solutions which are periodic of the form,

$$\psi = \psi_0(y) + \sum_{n=-\infty}^{\infty} f_n(y)\, e^{i\alpha n(ct-x)}, \tag{2}$$

with c real.

This solution represents a laminar flow $\bar{u}$ derivable from $(\psi_0 + f_0)$, on which is superimposed a disturbance which is periodic in x of wavelength $(2\pi/\alpha)$, but is of general shape, and which propagates in the x direction with phase-velocity c. The reality of c expresses the condition of 'neutral' equilibrium.

In order that ψ shall be real we must have

$$f_{-n}(y) = \bar{f}_n(y), \tag{3}$$

where the bar denotes the complex conjugate. For an earlier discussion of this problem see Meksyn & Stuart (1951).

We shall consider the case where ψ_0 represents the plane Poiseuille flow,

$$\psi_0 = y - \tfrac{1}{3}y^3 \quad (-1 < y < 1), \tag{4}$$

$$u_0 = \frac{\partial \psi_0}{\partial y} = 1 - y^2, \quad v_0 = \frac{\partial \psi_0}{\partial x} = 0. \tag{5}$$

For weak disturbances, when only f_1 need be considered, we have the Orr–Sommerfeld problem for the perturbation of plane Poiseuille flow. For disturbances of finite amplitude, $\dot{f}_0$ represents the modification of the mean flow caused by the disturbance.

The boundary conditions of the vanishing of both velocity components at the walls ($y = \pm 1$) require that

$$f_n(1) = f_n(-1) = \dot{f}_n(1) = \dot{f}_n(-1) = 0. \tag{6}$$

Substitution of (2) in (1) yields

$$\ddot{g}_n - \alpha^2 n^2 g_n + in\alpha R[(1-y^2-c)g_n + 2f_n] = i\alpha R \sum_{m=-\infty}^{\infty} [(n-m)f_{n-m}\dot{g}_m - m\dot{f}_{n-m}g_m], \tag{7}$$

where g_n is the vorticity amplitude given by

$$g_n = \ddot{f}_n - \alpha^2 n^2 f_n. \tag{8}$$

In the linearized case, when the quadratic terms on the right-hand side of (7) are neglected, this equation reduces to the Orr–Sommerfeld equation. Using (3), (7) can be transformed into (see Eckhaus 1965, p. 98; Watson 1960)

$$\begin{aligned} &\ddot{g}_n - \alpha^2 n^2 g_n + in\alpha R[(1-y^2-c+\dot{f}_0)g_n + (2-\dddot{f}_0)f_n] = i\alpha R K_n, \\ K_n &= \sum_{k=1}^{n-1} [kf_k\dot{g}_{n-k} - (n-k)\dot{f}_k g_{n-k}] \\ &\quad + \sum_{k=1}^{\infty} [-k\bar{f}_k\dot{g}_{n+k} - (n+k)\dot{\bar{f}}_k g_{n+k} + k\dot{f}_{n+k}\bar{g}_k + (n+k)f_{n+k}\dot{\bar{g}}_k] \quad (n \geqslant 1). \end{aligned} \tag{9}$$

The equation for f_0 reduces to

$$f_0^{\mathrm{IV}} = 2\alpha R \mathscr{I} \sum_{k=1}^{\infty} (k\dot{\bar{f}}_k g_k + k\bar{f}_k\dot{g}_k) = 2\alpha R \mathscr{I} \frac{d^2}{dy^2} \sum_{k=1}^{\infty} k\bar{f}_k\dot{f}_k, \tag{10}$$

where $\mathscr{I}$ denotes the imaginary part.

Equation (10) can be integrated into

$$\dddot{f}_0 = 2\alpha R \mathscr{I} \sum_{k=1}^{\infty} k\bar{f}_k\ddot{f}_k. \tag{11}$$

It can be shown (Stuart 1960, Watson 1960) that putting the constant of integration in (11) equal to zero is equivalent to making the assumption that the mean pressure-gradient remains unchanged by the perturbation. This would be realized in practice if the mean pressure is maintained constant both at the intake of the channel and at the outflow. A further integration of (11) leads to

$$\ddot{f}_0 = 2\alpha R \mathscr{I} \sum_{k=1}^{\infty} k\bar{f}_k\dot{f}_k, \tag{12}$$

where now the constant of integration has been put equal to zero because f_0 is an odd function of y.

The integral of (12) which satisfies the boundary condition $\dot{f}_0(1) = 0$ is

$$\dot{f}_0 = -2\alpha R \mathscr{I} \int_y^1 \sum_{k=1}^{\infty} k\bar{f}_k\dot{f}_k\, dy. \tag{13}$$

If we truncate the series (2) at $n = m$, then (9) represents a system of m ordinary non-linear differential equations for the f_n, with $\dot{f}_0$ and $\ddot{f}_0$ given by (13) and (11). Periodic solutions of finite amplitude will be demonstrated to exist if a solution of this system of equations can be found which satisfies the boundary conditions (6) for a real eigenvalue c, and if it can also be shown that the truncated series (2) has converged sufficiently well throughout the channel.

Method of solution

In order to solve the non-linear system of equations (9), we expand the solution in a power series of a parameter λ which is a measure of the magnitude of disturbance. Let

$$f_n(y) = \lambda^n F_n, \quad g_n(y) = \lambda^n G_n(y) \quad (1 \leqslant n \leqslant m). \tag{14}$$

Substitution of (14) in (9) shows that the appropriate perturbation-parameter to use is

$$\epsilon = 2\alpha R \lambda^2, \tag{15}$$

and we therefore adopt the expansions,

$$F_n(y) = \sum_{j=0}^{\sigma} \epsilon^j F_{nj}(y), \quad G_n(y) = \sum_{j=0}^{\sigma} \epsilon^j G_{nj}(y). \tag{16}$$

Similarly, we let

$$c = \sum_{j=0}^{\sigma} \epsilon^j c_j. \tag{17}$$

Here m and σ are truncation limits, and c_0 is the (complex) eigenvalue of the Orr–Sommerfeld equation for F_{10}

$$L_1(F_{10}) \equiv F_{10}^{\mathrm{IV}} - 2\alpha^2 \ddot{F}_{10} + \alpha^4 F_{10} + i\alpha R[(1-y^2-c_0)(\ddot{F}_{10} - \alpha^2 F_{10}) + 2F_{10}] = 0. \tag{18}$$

If, for given values of α and R, a value ϵ_1 is found which makes the imaginary part c_i of c in (17) vanish,

$$c_i = \sum_{j=0}^{\sigma} \epsilon_1^j c_{ji} = 0, \tag{19}$$

and if, furthermore, the series in (19) as well as the corresponding series in (16) show evidence of satisfactory convergence at $\epsilon = \epsilon_1$, then there exists a periodic undamped (neutral) solution for a finite amplitude λ_1. The latter is determined from ϵ_1 by (15).

Using (14), we write (9) in the modified form,

$$\begin{aligned}
&\ddot{G}_n - \alpha^2 n^2 G_n + i\alpha n R[(1-y^2-c_0) G_n + 2F_n] \\
&\quad = i\alpha n R[(c - c_0 - \dot{f}_0) G_n + \ddot{f}_0 F_n] + i\alpha R \left\{ \sum_{k=1}^{n-1} [k F_k \dot{G}_{n-k} - (n-k) \dot{F}_k G_{n-k}] \right. \\
&\qquad \left. + \sum_{k=1}^{m-n} (\epsilon/2\alpha R)^k [(n+k) F_{n+k} \dot{\bar{G}}_k + k \dot{F}_{n+k} \bar{G}_k - k \bar{F}_k \dot{G}_{n+k} - (n+k) \dot{\bar{F}}_k G_{n+k}] \right\}.
\end{aligned} \tag{20}$$

Substitution of the expansions (16) and (17) in (20) leads to a system of equations for the F_{nj}

$$\begin{aligned}
L_n(F_{nj}) &= F_{nj}^{\mathrm{IV}} - 2\alpha^2 n^2 \ddot{F}_{nj} + \alpha^4 n^4 F_{nj} + i\alpha n R[(1-y^2-c_0) G_{nj} + 2F_{nj}] \\
&= i\alpha n R[(c_j - E_j) G_{n0} + T_{nj} + Q_{nj}] + i\alpha R \Theta_{nj} \quad (j = 0, 1, \dots, \sigma),
\end{aligned} \tag{21}$$

where

$$\Theta_{nj} = \left\{\sum_{k=1}^{n-1}\sum_{s=0}^{j}[kF_{k,j-s}\dot{G}_{n-k,s}-(n-k)\dot{F}_{k,j-s}G_{n-k,s}]\right.$$
$$+\sum_{k=1}^{(j,\,m-n)}(2\alpha R)^{-k}\sum_{s=0}^{j-k}[(n+k)F_{n+k,j-k-s}\dot{\bar{G}}_{ks}+k\dot{F}_{n+k,j-k-s}\bar{G}_{ks}$$
$$\left.-k\bar{F}_{k,j-k-s}\dot{G}_{n+k,s}-(n+k)\dot{\bar{F}}_{k,j-k-s}G_{n+k,s}]\right\}. \qquad (22)$$

Here

$$T_{nj} = \sum_{s=1}^{j-1}(c_s-E_s)\,G_{n,j-s}, \qquad (23)$$

$$E_j = -\mathscr{I}\int_y^1 dy\sum_{k=1}^{(m,\,j)}k(2\alpha R)^{1-k}D_{k,j-k}, \quad D_{kj} = \sum_{s=0}^{j}\bar{F}_{ks}\dot{F}_{k,j-s}, \qquad (24)$$

$$Q_{nj} = \sum_{s=1}^{j}L_sF_{n,j-s}, \quad L_j = \mathscr{I}\sum_{k=1}^{(m,\,j)}k(2\alpha R)^{1-k}H_{k,j-k}, \qquad (25)$$

$$H_{kj} = \sum_{s=0}^{j}\ddot{F}_{ks}\bar{F}_{k,j-s}. \qquad (26)$$

With the above definitions we have

$$\dot{f}_0 = \sum_{j=1}^{\sigma}\epsilon^jE_j, \quad \ddot{f}_0 = \sum_{j=1}^{\sigma}\epsilon^jL_j. \qquad (27)$$

We note that for the case $n = 1$ in (21),

$$L_1(F_{1j}) = F_{1j}^{\mathrm{IV}} - 2\alpha^2\ddot{F}_{1j}+\alpha^4F_{1j}+i\alpha R[(1-y^2-c_0)(F_{1j}-\alpha^2F_{1j})+2F_{1j}]$$
$$= i\alpha R[(c_j-E_j)\,G_{10}+T_{1j}+Q_{1j}+\Theta_{1j}], \qquad (28)$$

the operator $L_1(F_{1j})$ is the Orr–Sommerfeld operator of (18), and c_0 is the appropriate eigenvalue corresponding to the boundary conditions (6). If $\tilde{F}_1$ is the solution of the adjoint operator to L_1,

$$\tilde{L}_1(\tilde{F}_1) = \tilde{F}_1^{\mathrm{IV}} - 2\alpha^2\ddot{\tilde{F}}+\alpha^4\tilde{F}_1+i\alpha R[(1-y^2-c_0)(\ddot{\tilde{F}}_1-\alpha^2\tilde{F}_1)-4y\dot{\tilde{F}}_1] = 0, \qquad (29)$$

$$\tilde{F}_1(1) = \dot{\tilde{F}}_1(1) = \tilde{F}_1(-1) = \dot{\tilde{F}}_1(-1) = 0, \qquad (30)$$

then we have

$$\int_{-1}^{1}L_1(F_{1j})\,\tilde{F}_1\,dy = 0. \qquad (31)$$

Applying (31) to (28), we get

$$\int_{-1}^{1}[(c_j-E_j)\,G_{10}+T_{1j}+Q_{1j}+\Theta_{1j}]\,\tilde{F}_1\,dy = 0. \qquad (32)$$

We shall adopt a normalization for $\tilde{F}_1$ given by

$$\int_{-1}^{1}G_{10}\tilde{F}_1\,dy = 1, \qquad (33)$$

whereby (32) yields

$$c_j = E_j - \int_{-1}^{1}(T_{1j}+Q_{1j}+\Theta_{1j})\,\tilde{F}_1\,dy \quad (j = 1,\ldots,\sigma). \qquad (34)$$

Since T_{1j} as defined in (23) contains c_j's only up to c_{j-1}, we can determine the c_j successively from (34).

The orthogonality condition (32) has to be imposed in order to make a solution of (28) possible at all. Even then, the solution of the inhomogeneous equation is arbitrary by a multiple of the solution of the homogeneous equation. This, however, does not affect the values of the c's (Pekeris 1936).

We have solved the system of (21) by finite differences, using a transformation due to Thomas (1953). Details of the method will be found in an earlier publication (Pekeris & Shkoller 1967).† In the finite-difference scheme, each differential equation is reduced to a system of $N = 1/h$ simultaneous inhomogeneous linear equations for the values of $F_{nj}(kh)$, where h denotes the finite-difference interval. The system is linear because the work can be so arranged that the inhomogeneous part is derived from earlier solutions.

When $n > 1$, the determinant Δ_n of the system of equations does not vanish because c_0 is not an eigenvalue for $n \neq 1$. The solution of the inhomogeneous system can then be carried out in a straightforward manner. For the case $n = 1$ given in equation (28), Δ_1 vanishes, and the solution takes on the indeterminate form of 0/0. Here the zero in the numerator is achieved by meeting the orthogonality condition (32). In a previous case (Pekeris 1936), where an analytical solution was sought, this difficulty was overcome, but in our case of a numerical solution we have followed the following device. Equation (28) was solved first using a value of $c_0(1+\delta)$ instead of c_0, and then again for $c_0(1-\delta)$, and the mean of the two solutions was adopted. Obviously we want to make δ as small as possible, in order to approach the limit as $\delta \to 0$. However, when δ is too small the solution becomes inaccurate because Δ_1 gets close to zero. We have found that $\delta = 5 \times 10^{-4}$ is about optimal, for which the relative deviation of the two solutions from the mean is of the order δ.

Since the above method is unconventional, we have checked the results by the following independent method. In order to solve (28),

$$L_1(F_{1j}) = F_{1j}^{\mathrm{IV}} - 2\alpha^2 \ddot{F}_{1j} + \alpha^4 F_{1j} + i\alpha R[(1-y^2-c_0)(\ddot{F}_{1j} - \alpha^2 F_{1j}) + 2F_{1j}] = i\alpha R K_{1j}, \tag{35}$$

where
$$K_{1j} = [(c_j - E_j) G_{10} + T_{1j} + Q_{1j} + \Theta_{1j}] \quad (j = 1, \ldots, \sigma), \tag{36}$$

we develop F_{1j} in terms of the eigenfunctions ϕ_1^ν of the Orr–Sommerfeld equation,

$$L_1(\phi_1^\nu) = \phi_1^{\nu\mathrm{IV}} - 2\alpha^2 \ddot{\phi}_1^\nu + \alpha^4 \phi_1^\nu + i\alpha R[(1-y^2-c_1^\nu)(\ddot{\phi}_1^\nu - \alpha^2 \phi_1^\nu) + 2\phi_1^\nu] = 0, \tag{37}$$

$$F_{ij}(y) = \sum_{\nu=2}^{\mu} A_{j\nu} \phi_1^\nu(y). \tag{38}$$

In (37), the first eigenvalue c_1^1 is identical with c_0 of (35), and in the summation (38) we omit the first eigenfunction ϕ_1^1, which is the solution of the homogeneous equation (35). Substituting (38) into (35), we get

$$\sum_{\nu=1}^{\mu} A_{j\nu}(c_1^\nu - c_0)(\ddot{\phi}_1^\nu - \alpha^2 \phi_1^\nu) = K_{1j}. \tag{39}$$

† In equation (52) of that paper, $(\omega_1 - P_1)$ should be $(\omega_1 + P_1)$. The numbers given in tables 1 and 2 are 2β, not β.

Using the orthogonality condition,

$$\int_1^1 (\ddot{\phi}_1^\nu - \alpha^2 \phi_1^\nu)\, \tilde{\phi}_1^\sigma \, dy = \delta_{\nu\sigma}, \tag{40}$$

where $\tilde{\phi}_1^\nu$ is the solution of (29) adjoint to (37), we can solve (39) for the expansion coefficients $A_{j\nu}$

$$A_{j\nu} = \frac{2}{(c_1^\nu - c_0)} \int_0^1 K_{1j}(y)\, \tilde{\phi}_1^\nu(y)\, dy. \tag{41}$$

We have solved (35) by (38) and (41) going up to values of μ above 50, and found the resulting coefficients c_j in the expansion (17), as well as the critical amplitude parameter λ^* in (44) infra, to be identical with the values obtained by the first method. As a further check (9) was solved directly, using the final vectors in the evaluation of K_n. The new solutions differed little from the input vectors. The eigenvalue c was evaluated from

$$c = c_0 + (1/\beta) \int_0^1 \tilde{f}_{10} (\dot{f}_0 g_1 - \dddot{f}_0 f_1 - K_1)\, dy; \quad \beta = \int_0^1 g_1 \tilde{f}_{10}\, dy, \tag{42}$$

and found to agree with the (real) input value.

Discussion of results

The expansion of the phase velocity c appearing in (2) into the power series (17) of the amplitude-parameter ϵ is related to Stuart's theory (Stuart 1958, 1960; also Watson 1960), based on a conjecture of Landau (Landau 1944) about periodic disturbances of finite amplitude of plane Poiseuille flow in the vicinity of the neutral curve. We have discussed this problem, as formulated by Eckhaus (1965), in a previous paper (Pekeris & Shkoller 1967). Indeed, it can be shown that the integral $\beta(\alpha, R)$, defined in (34) of that paper and evaluated in table 1 and table 2, is a multiple of c_1 in the expansion (17)

$$\beta = \beta_r + i\beta_i = -i\alpha^2 R c_1 = \alpha^2 R c_{1i} - i\alpha^2 R c_{1r}. \tag{43}$$

We have found satisfactory numerical agreement between our values of c_1 and the values of β referred to. Our previous analysis showed that c_{1i}, or β_i, is positive inside the dashed curve shown in figure 1, and is negative outside it. As a result, the neutral curve for finite amplitudes is shifted from the neutral curve ABC valid for infinitesimal disturbances, to the curve DBF. Here the branch DB lies in the zone which is stable for infinitesimal disturbances, while the branch BF lies inside the unstable zone. Along the branch BF, a disturbance which is unstable for $\lambda = 0$ becomes stable (or neutral) for some finite value λ of the amplitude of the disturbance, as was conjectured by Landau (1944). It should be pointed out that the curve DBF in figure 1 was drawn only schematically, in that actually the distance CF is much smaller than AD.

These qualitative results, which follow from the mere change of the sign of c_{1i} as the dashed line in figure 1 is crossed, are now confirmed quantitatively through our evaluation of the whole series-expansion (17). The results are shown in figure 2. Essentially, the curves in figure 2 show the regions in the $\alpha - R$ plane where the series (17) was found to converge. Convergence was manifest for $\epsilon < \frac{1}{3}$.

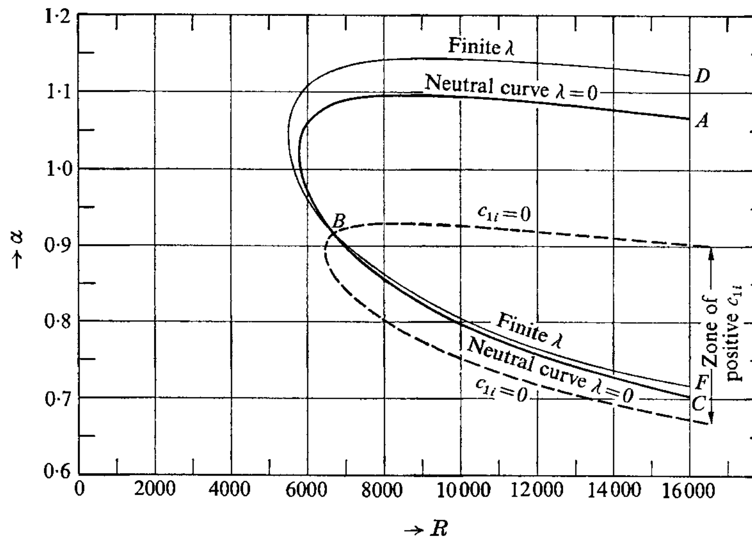

FIGURE 1. Schematic illustration of the shifting of the neutral curve for a finite amplitude λ of the disturbance of plane Poiseuille flow.

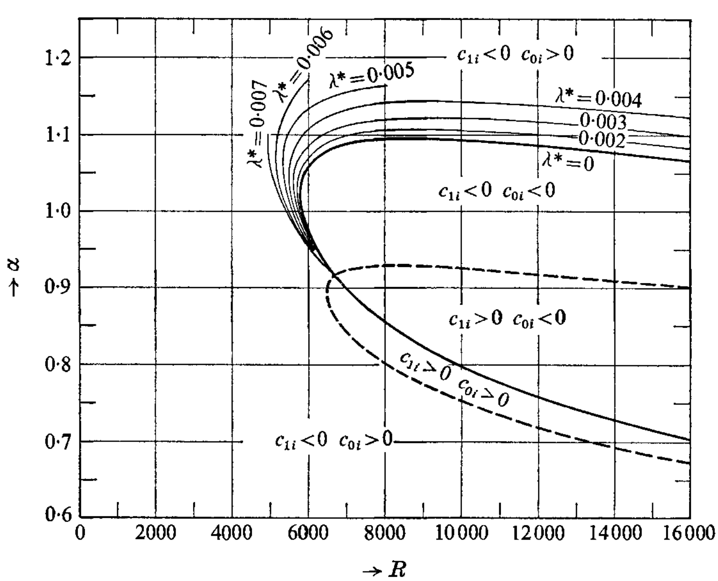

FIGURE 2. The neutral curves for finite amplitude disturbances of plane Poiseuille flow. λ^* is the amplitude parameter. The dashed curve is the locus where c_{1i} in equation (17) vanishes.

The amplitude parameter λ^* was defined through

$$\lambda^* = \left|\sum_{n=1}^{m} f_n(0)\right| = \left|\sum_{n=1}^{m} \lambda^n F_n(0)\right|. \tag{44}$$

$F_1(0)$ was taken to be equal to 1, and the values of the other $F_n(0)$ then followed from the solution of (21). λ^* is then the absolute value of the sum of the values of f_n on the axis of the channel. λ^* exceeded λ generally by from 5 to 8 %. The above definition of λ^* was adopted, in order to facilitate comparison of our results with those derived from another approach which we made for the case when the series in (2) was truncated at $n = 1$. Equation (9) then reduces to the non-linear eigenvalue problem:

$$f_1^{\mathrm{IV}} - 2\alpha^2 \ddot{f}_1 + \alpha^4 f_1 + i\alpha R[(1-y^2-c+\dot{f}_0)(f_1-\alpha^2 f_1) + (2-\dddot{f}_0)f_1] = 0, \tag{45}$$

$$\dot{f}_0 = -2\alpha R\mathscr{I}\int_y^1 \bar{f}_1 \dot{f}_1\, dy, \quad \dddot{f}_0 = 2\alpha R\mathscr{I}\bar{f}_1\ddot{f}_1, \tag{46}$$

with f_1 satisfying the boundary conditions (6). Starting with

$$f_1 = \lambda \quad \text{at} \quad y = 0, \tag{47}$$

where λ is initially very small, the system of (45) and (46) was iterated until a convergent numerical solution was obtained. The resulting c_i was different from zero. With this solution as a start, another one was then obtained by iteration for a higher value of λ. c_i was thus explored as a function of λ, and the value λ^* where c_i vanishes (for the given values of α and R) was determined. These values of λ^* were found to agree with those derived by the present method for the case $m = 1$, but with σ taken large enough to assure convergence.

The velocity-fluctuations in the range of parameters explored in figure 2 have a distribution across the channel which is close to that in the first eigenfunction of the Orr–Sommerfeld equation. This follows from the fact that the amplitude-parameter λ, as well as λ^*, is less than 1/100, so that the stream function ψ in (2) is, by (14), given approximately by

$$\begin{aligned}\psi &\simeq (\psi_0+f_0) + 2\mathscr{R}[f_1(y)\, e^{i\alpha(ct-x)}] \\ &\simeq (\psi_0+f_0) + 2\lambda\mathscr{R}[F_{10}(y)\, e^{i\alpha(ct-x)}].\end{aligned} \tag{48}$$

Here $F_{10}(y) = \phi_1^1(y)$ is the first even eigenfunction of the Orr–Sommerfeld equation, and $\mathscr{R}$ denotes the real part. The fluctuating velocities derivable from the term in brackets in (48) have therefore a distribution in y derived from $\phi_1^1(y)$, with an external amplitude-factor λ. The r.m.s. of the velocity fluctuations is therefore proportional to λ, which is within 8% of λ^*. In contrast to the fluctuating velocity field, the curvature of the mean flow is markedly modified, as discussed below.

The variation of λ^* with R for constant α is shown in figure 3 for the region lying above the branch AB of the neutral curve. In the region lying above the branch BC of the neutral curve the variation of λ^* with α is very steep, so that all the curves up to $\lambda^* = 0{\cdot}004$ lie very close to and above the neutral curve, and cannot be discerned in figure 2. This is exhibited by the steepness of the curves shown in figure 4.

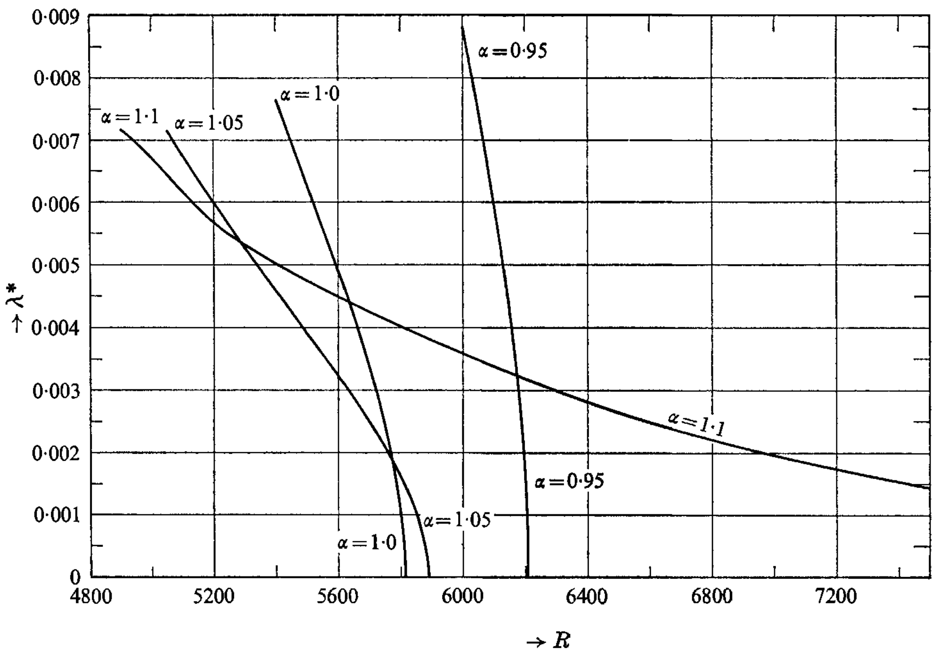

FIGURE 3. Variation of the amplitude parameter λ^* with R for constant α along the DB branch of figure 1.

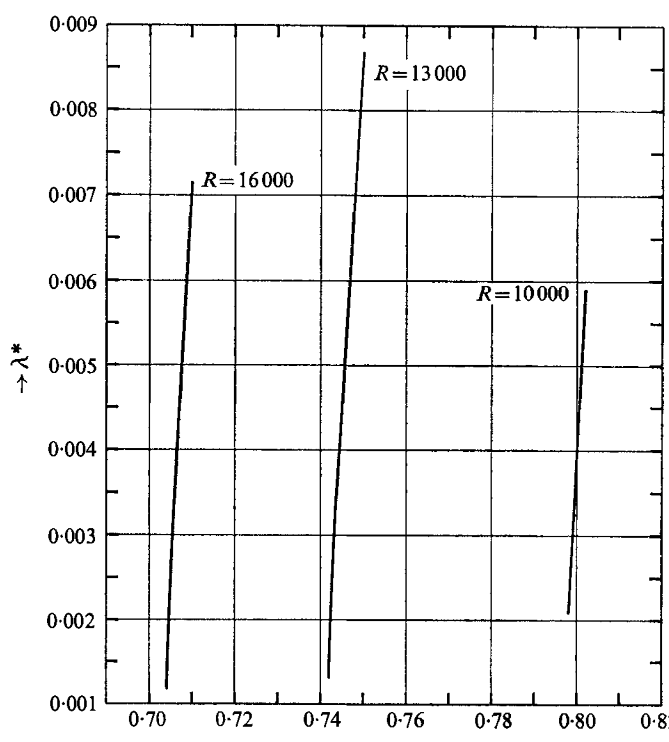

FIGURE 4. Variation of the amplitude parameter λ^* with α for constant R along the BF branch of figure 1.

In the case of periodic disturbances of finite amplitude the mean flow is modified by the terms $\dot{f}_0$ and $\ddot{f}_0$ appearing in (9). The original velocity profile $(1-y^2)$ is changed to $(1-y^2+\dot{f}_0)$. In the region of the $\alpha-R$ plane explored in figure 2 this change due to the $\dot{f}_0$ term is less than 1·5 %. The curvature of the velocity profile is changed from the value of 2 to $(2-\ddot{f}_0)$. It is known that when the curvature changes sign the velocity profile so modified becomes unstable even to infinitesimal disturbances. In figure 5 the curvature $(2-\ddot{f}_0)$ is plotted for the case of

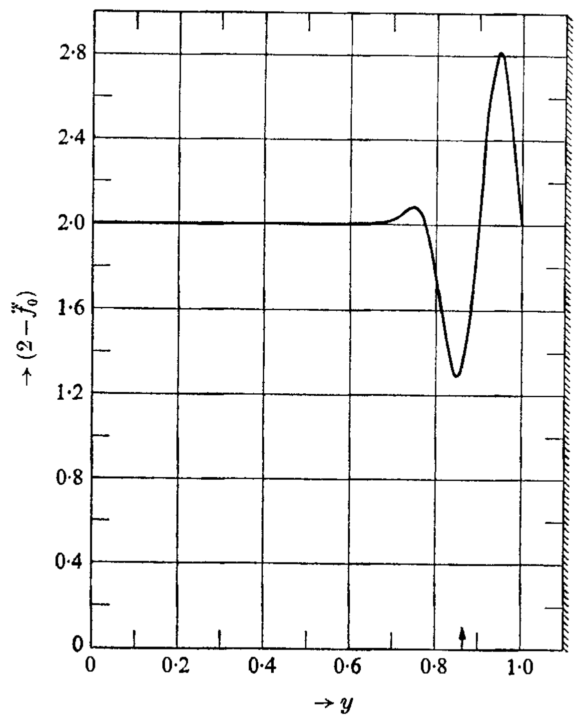

FIGURE 5. The curvature of the velocity profile $(2-\ddot{f}_0)$ for the case $R = 12{,}000$, $\alpha = 1{\cdot}14$, $\lambda^* = 0{\cdot}0043$, $c_r = 0{\cdot}2485$.

$R = 12{,}000$, $\alpha = 1{\cdot}14$, $\lambda^* = 0{\cdot}0043$. It is seen that $(2-\ddot{f}_0)$ is modified up to 40 % in the vicinity of the wall, showing a tendency to become negative as λ is increased further. The arrow indicates the position where the phase-velocity of the disturbance c_r equals the velocity $(1-y^2)$ in the Poiseuille flow. The minimum value of $(2-\ddot{f}_0)$ occurs just to the left of that position.

It is to be noted that, with increase in the amplitude λ of the disturbance, the phase velocity increases. The maximum change in the region explored in figure 2 is about 3 %. This suggests a qualitative similarity to the Riemann solution for acoustic waves of finite amplitude.

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